SOFT $\pi$-OPEN SETS IN SOFT GENERALIZED TOPOLOGICAL SPACES

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Abstract – The main purpose of this paper is to study some interesting properties of the soft mapping $\pi : S(U)_E \rightarrow S(U)_E$ which satisfy the condition $\pi F_B \subseteq \pi F_D$ whenever $F_B \subseteq F_D \subseteq F_E$. A new class of generalized soft open sets, called soft $\pi$-open sets is introduced and studied their basic properties. A soft set $F_G \subseteq F_E$ is said to be a soft $\pi$-open set iff $F_G \subseteq \pi F_G$. The notions of soft interior and soft closure are generalized using these sets. We then introduce the concepts of soft $\pi$-interior $i_{\pi} F_G$, soft $\pi$-closure $c_{\pi} F_G$, soft $\pi^* F_G$ of a soft set $F_G \subseteq F_E$. Under suitable conditions on $\pi$, the soft $\pi$-interior $i_{\pi} F_G$ and the soft $\pi$-closure $c_{\pi} F_G$ of a soft set $F_G \subseteq F_E$ are easily obtained by explicit formulas. The soft $\mu$-semi-open sets, soft $\mu$-pre-open sets, soft $\mu$-$\alpha$-open sets and soft $\mu$-$\beta$-open sets for a given Soft Generalized Topological Space $(F_E, \mu)$ can be obtained from soft $\pi$-open sets which are important for further research on soft generalized topology.

Keywords – Soft sets, soft generalized topology, soft mapping, soft $\pi$-open sets, soft $\pi$-interior, soft $\pi$-closure.

1 Introduction

The concept of soft set theory was introduced by Molodtsov [19] in 1999 as a mathematical tool for modeling uncertainties. Molodtsov successfully applied the soft set theory in several directions such as game theory, probability, Perron and Riemann Integration, theory of measurements [20]. Maji et al [17] and Naim Cagman et al. [5] have further modified the theory of soft sets which is similar to that of Molodtsov. After the introduction of the notion of soft sets, several researchers improved this concept. Cagman [6] presented the soft matrix theory and set up the maximum decision making method. D. Pei and D Miao [21] showed that soft sets are a class of special information systems. Babitha and Sunil [4] studied the soft set relation and discussed some related concepts. Kharal et al. [16] introduced soft functions over classes of soft sets. The notion of soft ideal is initiated for
the first time by Kandil et al. [13]. Feng et al. [9] worked on soft semi rings, soft ideals and idealistic soft semi rings.

It is known that topology is an important area of mathematics, with many applications in the domain of computer science and physical sciences. Topological structure of soft sets was also studied by many researchers. Shabir and Naz [22] and Cagman [7] initiated the study of soft topology and soft topological spaces independently. Shabir and Naz defined soft topology on the collection of soft sets over an initial universe with a fixed set of parameters. On the other hand, Cagman et al. [7] introduced soft topology on a soft set and defined soft topological space. The notion of soft topology by Cagman is more general than that by Shabir and Naz. B Ahmad and S Hussain [1] explored the structures of soft topology using soft points. Weak forms of soft open sets were first studied by Chen [8]. He investigated soft semi-open sets in soft topological spaces and studied some properties of it. Arockiarani and Lancy [3] are defined soft β-open sets and continued to study weak forms of soft open sets in soft topological space. Akdag and Ozkan [2], defined soft α-open and soft α-closed sets in soft topological spaces and studied many important results and some properties of it. Soft pre-open sets were introduced by [3]. Kandil et al. [14] introduced a unification of some types of different kinds of subsets of soft topological spaces using the notion of γ-operations. Kandil et al. [15] generalize this unification of types of different kinds of subsets of soft topological spaces using the notion of γ-oprations to supra topological spaces. Soft generalized topology is relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches that will significantly contribute to the solution of complex problems in engineering and environment. Jyothis and Sunil [10] introduced the notion of soft generalized topology (SGT) on a soft set and studied basic concepts of soft generalized topological spaces (SGTS). It is showed that a soft generalized topological space gives a parameterized family of generalized topological space. They also define and discuss the properties of soft generalized separation axioms which are important for further research on soft topology [12]. Jyothis and Sunil [11] introduced the concept of soft μ-compactness in soft generalized topological spaces as a generalization of compact spaces.

This paper is organized as follows. In section 2, we begin with the basic definitions and important results related to soft set theory which are useful for subsequent sections. In section 3, the definitions and basic theorems of soft generalized topology on an initial soft set are given. Finally in section 4, we study some interesting properties of the soft mapping π : S(U)_E → S(U)_E which satisfy the condition πF_B ⊂ πF_D whenever F_B ⊂ F_D ⊂ F_E. We introduce the concept of soft π-open sets and study their basic properties. The most important special cases are obtained if μ is a SGT, i_μ and c_μ denote the soft μ-interior and soft μ-closure respectively, and π = c_μ i_μ, π = i_μ c_μ, π = i_μ c_μ i_μ and π = c_μ i_μ c_μ. The corresponding soft π-open sets are called the soft μ-semi-open sets, soft μ-pre-open sets, soft μ-α-open sets and soft μ-β-open sets. Under suitable conditions on π, the soft π-interior i_π F_G and the soft π-closure c_π F_G of a soft set F_G ⊂ F_E are easily obtained by explicit formulas.

2 Preliminaries

In this section we recall some definitions and results defined and discussed in [5, 10, 11, 16]. Throughout this paper U denotes the initial universe, E denotes the set of all possible parameters, P(U) is the power set of U and A is a nonempty subset of E.
Definition 2.1. A soft set $F_A$ on the universe $U$ is defined by the set of ordered pairs $F_A = \{(e, f_A(e)) \mid e \in E, f_A(e) \in \mathcal{P}(U)\}$, where $f_A : E \to \mathcal{P}(U)$ such that $f_A(e) = \emptyset$ if $e \notin A$. Here $f_A$ is called an approximate function of the soft set $F_A$. The value of $f_A(e)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection. The set of all soft sets over $U$ with $E$ as the parameter set will be denoted by $S(U)_E$ or simply $S(U)$.

Definition 2.2. Let $F_A \in S(U)$. If $f_A(e) = \emptyset$ for all $e \in E$, then $F_A$ is called an empty soft set, denoted by $F_\emptyset$. $f_A(e) = \emptyset$ means that there is no element in $U$ related to the parameter $e$ in $E$. Therefore we do not display such elements in the soft sets as it is meaningless to consider such parameters.

Definition 2.3. Let $F_A \in S(U)$. If $f_A(e) = U$ for all $e \in A$, then $F_A$ is called an $A$-universal soft set, denoted by $F^\sim_A$. If $A = E$, then the $A$-universal soft set is called an universal soft set, denoted by $F^\sim_E$.

Definition 2.4. Let $F_A, F_B \in S(U)$. Then $F_B$ is a soft subset of $F_A$ (or $F_A$ is a soft superset of $F_B$), denoted by $F_B \subseteq F_A$, if $f_B(e) \subseteq f_A(e)$, for all $e \in E$.

Definition 2.5. Let $F_A, F_B \in S(U)$. Then $F_B$ and $F_A$ are soft equal, denoted by $F_B = F_A$, if $f_B(e) = f_A(e)$, for all $e \in E$.

Definition 2.6. Let $F_A, F_B \in S(U)$. Then, the soft union of $F_A$ and $F_B$, denoted by $F_A \cup F_B$, is defined by the approximate function $f_{A\cup B}(e) = f_A(e) \cup f_B(e)$.

Definition 2.7. Let $F_A, F_B \in S(U)$. Then, the soft intersection of $F_A$ and $F_B$, denoted by $F_A \cap F_B$, is defined by the approximate function $f_{A\cap B}(e) = f_A(e) \cap f_B(e)$.

Definition 2.8. Let $F_A, F_B \in S(U)$. Then, the soft difference of $F_A$ and $F_B$, denoted by $F_A \setminus F_B$, is defined by the approximate function $f_{A\setminus B}(e) = f_A(e) \setminus f_B(e)$.

Definition 2.9. Let $F_A \in S(U)$. Then, the soft complement of $F_A$, denoted by $(F_A)^c$, is defined by the approximate function $f_{(F_A)^c}(e) = (f_A(e))^c$, where $(f_A(e))^c$ is the complement of the set $f_A(e)$, that is, $(f_A(e))^c = U \setminus f_A(e)$ for all $e \in E$.

Clearly $((F_A)^c)^c = F_A$, $(F_\emptyset)^c = F_E$, and $(F^\sim_E)^c = F_\emptyset$.

Definition 2.10. Let $F_A \in S(U)$. The soft power set of $F_A$, denoted by $\mathcal{P}(F_A)$, is defined by $\mathcal{P}(F_A) = \{F_{A_i} \mid F_{A_i} \subseteq F_A, i \in I \subseteq N\}$.

Theorem 2.11. Let $F_A, F_B, F_C \in S(U)$. Then,

1. $F_A \cup F_A = F_A$.
2. $F_A \cap F_A = F_A$.
3. $F_A \cup F_\emptyset = F_A$.
4. $F_A \cap F_\emptyset = F_\emptyset$.
5. $F_A \cup F^\sim_E = F^\sim_E$.
6. $F_A \cap F^\sim_E = F^\sim_A$.
7. $F_A \cup (F_A)^c = F^\sim_E$.
8. $F_A \cap (F_A)^c = F_\emptyset$.
9. $F_A \cup F_B = F_B \cup F_A$. 

(10) \( F_A \cap F_B = F_B \cap F_A \).
(11) \( (F_A \cup F_B)^c = (F_A)^c \cup (F_B)^c \).
(12) \( (F_A \cap F_B)^c = (F_A)^c \cup (F_B)^c \).
(13) \( (F_A \cup F_B) \cap F_C = F_A \cup (F_B \cap F_C) \).
(14) \( (F_A \cap F_B) \cap F_C = F_A \cap (F_B \cap F_C) \).
(15) \( F_A \cup (F_B \cap F_C) = (F_A \cup F_B) \cap (F_A \cup F_C) \).
(16) \( F_A \cap (F_B \cup F_C) = (F_A \cap F_B) \cup (F_A \cap F_C) \).

**Definition 2.12.** [16] Let \( S(U)_E \) and \( S(V)_K \) be the families of all soft sets over \( U \) and \( V \), respectively. Let \( \varphi : U \to V \) and \( \chi : E \to K \) be two mappings. The soft mapping
\[
\varphi \chi : S(U)_E \to S(V)_K
\]
is defined as:

(1) Let \( F_A \) be a soft set in \( S(U)_E \). The image of \( F_A \) under the mapping \( \varphi \chi \) is the soft set over \( V \), denoted by \( \varphi \chi (F_A) \) and is defined by
\[
\varphi \chi (f_A)(k) = \begin{cases} 
\bigcup_{e \in \chi^{-1}(k) \cap A} \varphi(f_A(e)), & \text{if } \chi^{-1}(k) \cap A \neq \emptyset; \\
\emptyset, & \text{otherwise}
\end{cases}
\]
for all \( k \in K \).

(2) Let \( G_B \) be a soft set in \( S(V)_K \). The inverse image of \( G_B \) under the mapping \( \varphi \chi \) is the soft set over \( U \), denoted by \( \varphi \chi^{-1}(G_B) \) and is defined by
\[
\varphi \chi^{-1}(g_B)(e) = \begin{cases} 
\varphi^{-1}(g_B(\chi(e))), & \text{if } \chi(e) \in B; \\
\emptyset, & \text{otherwise}
\end{cases}
\]
for all \( e \in E \).

The soft mapping \( \varphi \chi \) is called injective, if \( \varphi \) and \( \chi \) are injective. The soft mapping \( \varphi \chi \) is called surjective, if \( \varphi \) and \( \chi \) are surjective.

The soft mapping from \( S(U)_E \) to itself is denoted by \( \varphi : S(U)_E \to S(U)_E \)

**Definition 2.13.** Let \( \varphi \chi : S(U)_E \to S(V)_K \) and \( \tau_\sigma : S(V)_K \to S(W)_L \), then the soft composition of the soft mappings \( \varphi \chi \) and \( \tau_\sigma \), denoted by \( \tau_\sigma \circ \varphi \chi \), is defined by \( \tau_\sigma \circ \varphi \chi = (\tau \circ \varphi)(\sigma \circ \chi) \).

### 3 Soft Generalized Topological Spaces

**Definition 3.1.** [10] Let \( F_A \in S(U) \). A Soft Generalized Topology (SGT) on \( F_A \), denoted by \( \mu \) or \( \mu_{F_A} \) is a collection of soft subsets of \( F_A \) having the following properties:

(1) \( F_\emptyset \in \mu \)
(2) Any soft union of members of \( \mu \) belongs to \( \mu \).
The pair \((F_A, \mu)\) is called a Soft Generalized Topological Space (SGTS).
Observe that \(F_A \in \mu\) must not hold.

**Definition 3.2.** [10] A soft generalized topology \(\mu\) on \(F_A\) is said to be strong if \(F_A \in \mu\).

**Definition 3.3.** [10] Let \((F_A, \mu)\) be a SGTS. Then, every element of \(\mu\) is called a soft \(\mu\)-open set.

**Definition 3.4.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then the collection \(\mu_{F_B} = \{F_D \cap F_B / F_D \in \mu\}\) is called a Subspace Soft Generalized Topology (SSGT) on \(F_B\). The pair \((F_B, \mu_{F_B})\) is called a Soft Generalized Topological Subspace (SGTSS) of \(F_A\).

**Definition 3.5.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then the soft \(\mu\)-interior of \(F_B\) denoted by \(i_\mu(F_B)\) is defined as the soft union of all soft \(\mu\)-open subsets of \(F_B\).

Note that \(i_\mu(F_B)\) is the largest soft \(\mu\)-open set that is contained in \(F_B\).

**Theorem 3.6.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then \(F_B\) is a soft \(\mu\)-open set if and only if \(F_B = i_\mu(F_B)\).

**Theorem 3.7.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_G, F_H \subseteq F_A\). Then

\[
\begin{align*}
1) & \quad i_\mu(i_\mu(F_G)) = i_\mu(F_G) \\
2) & \quad F_G \subseteq F_H \Rightarrow i_\mu(F_G) \subseteq i_\mu(F_H) \\
3) & \quad i_\mu(F_G) \cap i_\mu(F_H) \supseteq i_\mu(F_G \cap F_H) \\
4) & \quad i_\mu(F_G) \cup i_\mu(F_H) \subseteq i_\mu(F_G \cup F_H) \\
5) & \quad F_G \subseteq F_G.
\end{align*}
\]

**Definition 3.8.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then \(F_B\) is said to be a soft \(\mu\)-closed set if its soft complement \((F_B)^c\) is a soft \(\mu\)-open set.

**Theorem 3.9.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then the following conditions hold:

1) The universal soft set \(F_B\) is soft \(\mu\)-closed.
2) Arbitrary soft intersections of the soft \(\mu\)-closed sets are soft \(\mu\)-closed.

**Definition 3.10.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then the soft \(\mu\)-closure of \(F_B\), denoted by \(c_\mu(F_B)\) is defined as the soft intersection of all soft \(\mu\)-closed super sets of \(F_B\).

Note that \(c_\mu(F_B)\) is the smallest soft \(\mu\)-closed superset of \(F_B\).

**Theorem 3.11.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). \(F_B\) is a soft \(\mu\)-closed set if and only if \(F_B = c_\mu(F_B)\).

**Theorem 3.12.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_B \subseteq F_A\). Then \(i_\mu(F_B) \subseteq F_B \subseteq c_\mu(F_B)\).

**Theorem 3.13.** [10] Let \((F_A, \mu)\) be a SGTS and \(F_G, F_H \subseteq F_A\). Then

\[
\begin{align*}
1) & \quad c_\mu(c_\mu(F_G)) = c_\mu(F_G) \\
2) & \quad F_G \subseteq F_H \Rightarrow c_\mu(F_G) \subseteq c_\mu(F_H) \\
3) & \quad c_\mu(F_G) \cap c_\mu(F_H) \supseteq c_\mu(F_G \cap F_H).
\end{align*}
\]
4 Soft π-Open Sets

Consider the soft mapping \( \pi : S(\mathcal{U})_E \rightarrow S(\mathcal{U})_E \) possessing the property of monotony, i.e., \( F_B \subseteq F_D \) imply \( \pi F_B \subseteq \pi F_D \). We denote the collection of all soft mapping having this property by \( \Pi \). Consider the following conditions for a soft mapping \( \pi \in \Pi \), \( F_B \subseteq F(\Pi_0) \)

\( \pi F_\emptyset = F_\emptyset \) \hspace{1cm} (Π0)

\( \pi F_E = F_E \) \hspace{1cm} (Π1)

\( \pi^2 F_B = \pi \pi F_B = \pi F_B \) \hspace{1cm} (Π2)

\( F_B \subseteq \pi F_B \) \hspace{1cm} (Π3)

\( \pi F_B \subseteq F_B \) \hspace{1cm} (Π4)

\( \pi^2 F_B \subseteq \pi F_B \) \hspace{1cm} (Π5)

Example 4.1. The soft identity mapping \( \text{id} : S(\mathcal{U})_E \rightarrow S(\mathcal{U})_E \) is in \( \Pi_0 \), \( \Pi_1 \), \( \Pi_2 \), \( \Pi_3 \), \( \Pi_4 \).

Let \( (F_E, \mu) \) be a SGTS and \( \text{i}_\mu : S(\mathcal{U})_E \rightarrow S(\mathcal{U})_E \) and \( \text{c}_\mu : S(\mathcal{U})_E \rightarrow S(\mathcal{U})_E \) be the soft \( \mu \)-interior and soft \( \mu \)-closure operators respectively. If \( \pi = \text{i}_\mu \), then \( \pi \in \Pi_0 \), \( \Pi_2 \), \( \Pi_4 \). If \( \pi = \text{c}_\mu \), then \( \pi \in \Pi_1 \), \( \Pi_2 \), \( \Pi_3 \).

Definition 4.2. A soft set \( F_G \subseteq F_E \) is said to be a soft π-open set iff \( F_G \subseteq \pi F_G \).

Example 4.3. The following are some examples of soft π-open sets:

1. \( F_\emptyset \) is always soft π-open for any \( \pi \in \Pi \)
2. \( F_E \) is soft π-open iff \( \pi \in \Pi_1 \)
3. Every soft set of the form \( \pi F_G \) is soft π-open if \( \pi \in \Pi_2 \)
4. Every soft subset of \( F_E \) is soft π-open if \( \pi \in \Pi_3 \)
5. If \( \pi \in \Pi_4 \), then \( F_G \) is soft π-open iff \( F_G = \pi F_G \)

Note: Let \( (F_E, \mu) \) be a SGTS. Then \( F_G \) is soft \( i_\mu \)-open (i.e., if \( \pi = i_\mu \)) iff \( F_G \subseteq i_\mu F_G \). But \( i_\mu F_G \subseteq F_G \). Thus \( F_G \) is soft \( i_\mu \)-open iff \( F_G = i_\mu F_G \) iff \( F_G \) is soft \( \mu \)-open by theorem 3.6. Hence soft \( i_\mu \)-open set coincides with the soft \( \mu \)-open sets.

Theorem 4.4. Any soft union of soft π-open sets is soft π-open.

Proof. Let \( \{F_{B_j}\}_{j \in J} \) be a collection of soft π-open sets. i.e., \( F_{B_j} \subseteq \pi F_{B_j} \forall j \in J \). Let \( F_B = \bigcup_{j \in J} F_{B_j} \). Now \( F_{B_j} \subseteq F_B \) imply \( \pi F_{B_j} \subseteq \pi F_B \forall j \in J \). Therefore \( F_B = \bigcup_{j \in J} F_{B_j} \subseteq \bigcup_{j \in J} \pi F_{B_j} \subseteq \pi F_B \). i.e., \( F_B \subseteq \pi F_B \). Hence \( F_B \) is soft π-open. ☐

Theorem 4.5. The collection of all soft π-open sets is a SGT.

Theorem 4.6. If \( \mu \) is a SGT on \( F_E \), then there is a soft mapping \( \pi \in \Pi_0 \), \( \Pi_2 \), \( \Pi_4 \) such that \( \mu \) is the collection of all soft π-open sets.
**Proof.** Define $\pi F_G$ to be the soft union of all $F_H \in \mu$ satisfying $F_H \subset F_G$. Then clearly $\pi F_G \in \mu$ and $\pi F_G \subset F_G$. $\pi F_G = F_G$. Now $F_H \in \mu \Rightarrow \pi F_H = F_H \supset F_H$ so that the elements of $\mu$ are soft $\pi$-open, while $F_G \subset \pi F_G \Rightarrow \pi F_G = F_G$ and $F_G \in \mu$. Finally $\pi F_G \in \mu \Rightarrow \pi \pi F_G = \pi F_G$.

**Definition 4.7.** Let $F_B \subset F_E$. The soft union of all soft $\pi$-open subsets of the soft set $F_B$ is called the soft $\pi$-interior of $F_B$, and is denoted by $i_\pi F_B$.

**Theorem 4.8.** The soft set $i_\pi F_B$ is the largest soft $\pi$-open subset of $F_B$.

Note: Let $(F_E, \mu)$ be a SGTS and suppose $\pi = i_\mu$, then the soft set $i_\mu F_B$ is the largest soft $i_\mu$-open subset of $F_B$. Since soft $i_\mu$-open sets are soft $\mu$-open sets, $i_\mu F_B$ is the largest soft $\mu$-open subset of $F_B$. Hence $i_\mu = i_\mu$.

**Theorem 4.9.** For any $\pi \in \Pi$ and $F_B \subset F_E$,

1. $i_\pi F_\emptyset = F_\emptyset$
2. $i_\pi F_B = i_{i_\pi F_B}$
3. $i_\pi F_B \subset F_B$, and
4. $i_\pi F_E = F_E$ iff $\pi F_E = F_E$

i.e., $i_\pi \in (\Pi 0), (\Pi 2), (\Pi 4)$ for any $\pi \in \Pi$; $i_\pi \in (\Pi 1)$ iff $\pi \in (\Pi 1)$

Conversely if $\pi \in (\Pi 0), (\Pi 2)$ and $(\Pi 4)$, then $\pi = i_\pi$.

**Proof.** First show that $i_\pi$ possess the property of monotony. Suppose $F_G \subset F_H$. By definition of $i_\pi$ and by theorem 4.8, $i_\pi F_G \subset F_G$ and $i_\pi F_H \subset F_H$. $i_\pi F_H$ is the largest soft $\pi$-open subset of $F_H$. Hence $i_\pi F_G \subset i_\pi F_H$. Clearly $i_\pi F_\emptyset = F_\emptyset$. i.e., $i_\pi \in (\Pi 0)$. By definition 4.7, $i_\pi F_G \subset F_G$ for any $F_G \subset F_E$, i.e., $i_\pi \in (\Pi 4)$. By theorem 4.8, $i_\pi F_G$ is soft $\pi$-open. so $i_\pi (i_\pi F_G) = \text{largest soft } \pi$-open subset of $i_\pi F_G = i_\pi F_G$. i.e., $i_\pi \in (\Pi 2)$. Again $i_\pi F_E = \text{largest soft } \pi$-open subset of $F_E = F_E$ $\iff F_E$ is a soft $\pi$-open set $\iff \pi \in (\Pi 1)$.

Conversely, assume that $\pi \in (\Pi 0), (\Pi 2)$ and $(\Pi 4)$, $\pi \in (\Pi 2) \Rightarrow \pi (\pi F_G) = \pi F_G \Rightarrow \pi F_G$ is soft $\pi$-open. $\pi \in (\Pi 4) \Rightarrow \pi F_G \subset F_G$ for any $F_G \subset F_E$. Therefore $\pi F_G$ is a soft $\pi$-open subset of $F_G$. Next if $F_H \subset F_G$ is soft $\pi$-open, then $F_H \subset \pi F_H \subset \pi F_G$. So $\pi F_G$ is largest soft $\pi$-open subset of $F_G$. Hence $i_\pi = \pi$.

**Theorem 4.10.** A soft set $F_G$ is soft $i_\pi$-open iff $F_G = i_\pi F_G$ iff $F_G$ is soft $\pi$-open.

**Proof.** $i_\pi$ possess the property of monotony. i.e., if $F_G \subset F_H$, then $i_\pi F_G \subset i_\pi F_H$. Also $i_\pi F_G \subset F_G$ for any $F_G \subset F_E$. Now $F_G$ is soft $i_\pi$-open iff $F_G \subset i_\pi F_G$ iff $F_G = i_\pi F_G$ iff $F_G$ is soft $\pi$-open by theorem 4.8.

**Definition 4.11.** A soft set $F_G \subset F_E$ is soft $\pi$-closed iff its soft complement $(F_G)^c$ is soft $\pi$-open.

Note: 1) Since $F_\emptyset$ is always soft $\pi$-open, $F_E$ is always soft $\pi$-closed, for any $\pi \in \Pi$

2) $F_G$ is soft $\pi$-closed iff $F_E$ is soft $\pi$-open iff $\pi \in (\Pi 1)$

3) If $\pi \in (\Pi 3)$, every soft subset of $F_E$ is soft $\pi$-closed.

**Theorem 4.12.** Any soft intersection of soft $\pi$-closed sets is soft $\pi$-closed.
Proof. Suppose \( \{ F_G \}_j \) be a collection of soft \( \pi \)-closed sets. Then \( \{ (F_G)^c \}_j \) is a collection of soft \( \pi \)-open sets. By theorem 4.4, \( \bigcup_{j \in J} (F_G)^c \) is soft \( \pi \)-open \( \Rightarrow (\bigcap_{j \in J} F_G)^c \) is soft \( \pi \)-closed. \( \blacksquare \)

**Theorem 4.13.** Let \( \xi \) be the collection of all soft \( \pi \)-closed sets. Then the following conditions hold.

1. The universal soft set \( F_E \in \xi \).
2. Arbitrary soft intersection of members of \( \xi \) belongs to \( \xi \).

**Definition 4.14.** The soft intersection of all soft \( \pi \)-closed supersets of \( F_G \) is called the soft \( \pi \)-closure of \( F_G \) and is denoted by \( c_\pi F_G \).

**Theorem 4.15.** The soft set \( c_\pi F_G \) is the smallest soft \( \pi \)-closed super set of \( F_G \).

Note: Let \((F_E, \mu)\) be a SGTS and if \( \pi = i_\mu \), then \( F_G \) is soft \( i_\mu \)-closed set \( \Leftrightarrow (F_G)^c \) is soft \( i_\mu \)-open \( \Leftrightarrow (F_G)^c \) is soft \( \mu \)-open \( \Leftrightarrow F_G \) is soft \( \mu \)-closed. Hence soft \( i_\mu \)-closed sets coincides with the soft \( \mu \)-closed ones and \( c_\mu = c_i \)

**Definition 4.16.** For any \( \pi \in \Pi \) and \( F_G \subset F_E \), \( \pi^* F_G = [\pi(F_G)^c]^c \).

**Theorem 4.17.** For any \( \pi \in \Pi \), the following conditions hold:

\[ \pi^* \in \Pi, (\pi^*)^* = \pi, \pi \in (\Pi 0) \Leftrightarrow \pi^* \in (\Pi 1), \pi \in (\Pi 1) \Leftrightarrow \pi^* \in (\Pi 0), \pi \in (\Pi 2) \Leftrightarrow \pi^* \in (\Pi 2), \pi \in (\Pi 3) \Leftrightarrow \pi^* \in (\Pi 4), (i_\pi)^* = c_\pi. \]

**Proof.** Assume that \( \pi \in \Pi \), i.e, if \( F_G \subset F_H \), then \( \pi F_G \subset \pi F_H \). Now \( F_G \subset F_H \Rightarrow (F_G)^c \supset (F_H)^c \Rightarrow (\pi F_G)^c \supset (\pi F_H)^c \Rightarrow (\pi F_G)^c \subset (\pi F_H)^c \Rightarrow \pi^* F_G \subset \pi^* F_H \). Hence \( \pi^* \in \Pi \). \( \pi^* F_G = (\pi(F_G))^c \). \( \therefore (\pi^*)^* F_G = [\pi^*(F_G)^c]^c = [\pi(F_G)^c]^c = \pi F_G \). Hence \( (\pi^*)^* = \pi \). \( \pi \in (\Pi 0) \Leftrightarrow \pi F_G = F_G \) \( \Leftrightarrow (\pi F_G)^c = F_E \Leftrightarrow (\pi F_G)^c = F_E \Leftrightarrow \pi^* F_G = F_G \Leftrightarrow \pi^* \in (\Pi 0), \pi \in (\Pi 1) \Leftrightarrow \pi F_E = F_E \Leftrightarrow (\pi F_E)^c = F_G \Leftrightarrow (\pi F_E)^c = F_G \Leftrightarrow \pi^* F_G = F_G \Leftrightarrow \pi^* \in (\Pi 0), \pi \in (\Pi 2) \Leftrightarrow \pi (\pi F_G)^c = \pi (F_G)^c \Leftrightarrow \pi (\pi F_G)^c \Leftrightarrow [\pi(\pi(F_G)^c)]^c \Leftrightarrow [\pi F_G]^c \Leftrightarrow \pi^* F_G \Leftrightarrow \pi^* F_G \Leftrightarrow \pi^* \in (\Pi 2). \pi \in (\Pi 3) \Leftrightarrow (F_G)^c \subset (\pi F_G)^c \Leftrightarrow (\pi F_G)^c \subset (\pi F_G)^c \Leftrightarrow (\pi F_G)^c \subset (\pi F_G)^c \Leftrightarrow (\pi F_G)^c \subset (\pi F_G)^c \Leftrightarrow \pi^* \in (\Pi 4). (i_\pi)^* F_G = (i_\pi(F_G))^c \). By theorem 4.8, \( i_\pi(F_G)^c \) is the largest soft \( \pi \)-open subset of \( (F_G)^c \). Hence its soft complement coincides with the smallest soft \( \pi \)-closed super set of \( F_G \), i.e, \( (i_\pi)^* F_G = c_\pi F_G \) for any \( F_G \subset F_E \). Hence \( (i_\pi)^* = c_\pi. \)

**Theorem 4.18.** Let \( (F_E, \mu) \) be a SGTS. Then \( (i_\mu)^* = c_\mu. \)

**Proof.** Take \( \pi = i_\mu \) and since \( i_{i\mu} = i_\mu \), the proof follows from theorem 4.17. \( \blacksquare \)

**Theorem 4.19.** A soft set \( F_G \subset F_E \) is soft \( \pi^* \)-closed \( \Leftrightarrow F_G \subset F_G \).

**Proof.** \( F_G \) is soft \( \pi^* \)-closed \( \Leftrightarrow (F_G)^c \) is soft \( \pi^* \)-open \( \Leftrightarrow (F_G)^c \subset \pi^*(F_G)^c \Leftrightarrow (F_G)^c \subset (\pi F_G)^c \Leftrightarrow \pi F_G \subset F_G. \) \( \blacksquare \)

**Theorem 4.20.** For any \( \pi \in \Pi \), \( c_\pi \in (\Pi 1), (\Pi 2), (\Pi 3); c_\pi \in (\Pi 0) \) iff \( \pi \in (\Pi 1) \). Conversely, if \( \pi \in (\Pi 1), (\Pi 2), (\Pi 3) \), then \( \pi = c_\pi. \)

**Proof.** Assume that \( F_G \subset F_H \subset F_E \). By theorem 4.15, \( c_\pi F_H \) is the smallest soft \( \pi \)-closed super set of \( F_H \). But \( F_H \supset F_G \). \( \therefore c_\pi F_H \) is a soft \( \pi \)-closed super set of \( F_G \). Again by theorem
4.15, $c_\pi F_G$ is the smallest soft $\pi$-closed super set of $F_G$. Hence $c_\pi F_H \supset c_\pi F_G \supset c_\pi \in \Pi$. Since $F_\varepsilon$ is a soft $\pi$-closed set, $c_\pi F_\varepsilon = F_\varepsilon \implies c_\pi \in (\Pi_1)$. By theorem 4.15, $c_\pi F_G$ is soft $\pi$-closed for any $F_G \subseteq F_\varepsilon$. Therefore $c_\pi (c_\pi F_G) = c_\pi F_G \implies c_\pi \in (\Pi_2)$. By theorem 4.15, $c_\pi F_G$ is the smallest soft $\pi$-closed super set of $F_G$, $c_\pi F_G \supset F_G \implies c_\pi \in (\Pi_3)$. $c_\pi F_G = F_\varnothing \iff F_\varnothing$ is soft $\pi$-closed set $\iff F_\varepsilon$ is soft $\pi$-open set $\iff F_\varepsilon = \pi F_\varepsilon$. Hence $c_\pi \in (\Pi_0)$ iff $\pi \in (\Pi_1)$.

Conversely, assume that $\pi \in (\Pi_1), (\Pi_2), (\Pi_3)$. Since $\pi \in (\Pi_2)$, $\pi(\pi F_G) = \pi F_G \iff \pi F_G$ is soft $\pi^*$-closed by theorem 4.19. Since $\pi \in (\Pi_3)$, $\pi F_G$ is soft $\pi^*$-closed super set of $F_G$, for any $F_G \subseteq F_\varepsilon$. If $F_H \supset F_G$ is a soft $\pi^*$-closed set, then by theorem 4.19, $\pi F_H \subset F_H$, so $F_H \supset \pi F_H \supset \pi F_G \supset F_G$. i.e., $\pi F_G$ is the smallest soft $\pi^*$-closed super set of $F_G$. Hence $\pi = c_\pi$.

**Theorem 4.21.** Any soft set $F_G$ is soft $i_\pi$-closed iff $F_G = c_\pi F_G$ iff $F_G$ is soft $\pi$-closed.

**Proof.** By theorem 4.9, $i_\pi \in \Pi$. By theorem 4.17 and 4.19, $F_G$ is soft $i_\pi$-closed $\iff F_G$ is soft $(i_\pi)^*$-closed $\iff F_G$ is soft $(c_\pi)^*$-closed $\iff c_\pi F_G \subset F_G \iff c_\pi F_G = F_G \iff F_G$ is soft $\pi$-closed by theorem 4.15.

**Theorem 4.22.** If $\pi_1, \pi_2 \in \Pi$, $\pi_2 \pi_1 \in \Pi$. If $\pi_1$ and $\pi_2 \in (\Pi_0), (\Pi_1), (\Pi_3), (\Pi_4)$, then $\pi_2 \pi_1 \in (\Pi_0), (\Pi_1), (\Pi_3), (\Pi_4)$ and $(\pi_2 \pi_1)^* = \pi_2^* \pi_1^*$.

Suppose the soft mappings $\theta, \sigma \in (\Pi_2)$. We will consider the soft mappings $\pi$ that are the products of factors $\theta$ or $\sigma$. Only the products of alternating factors $\theta, \sigma$ need be taken into consideration.

**Theorem 4.23.** If $\theta, \sigma \in (\Pi_2), 0 \sigma F_G \subset \sigma F_G, 0 \theta F_G \subset \sigma 0 \theta F_G$, and $0 F_G \subset \sigma 0 F_G$ for any $F_G \subset F_\varepsilon$. Then $\pi \in (\Pi_2)$ if $\pi$ is a product of alternating factors $\theta$ and $\sigma$.

**Proof.** Clearly $\pi \in \Pi$ by theorem 4.22. Since $0 \sigma F_G \subset \sigma F_G, 0 \theta 0 F_G \subset \sigma 0 F_G$ and hence $0 \theta 0 \sigma (F_G) \subset \sigma 0 F_G$. Again since $0 \sigma F_G \subset \sigma F_G$ and $0 F_G \subset \sigma 0 F_G \iff 0 F_G \subset 0 \sigma F_G \implies 0 \sigma 0 F_G \subset \sigma 0 F_G$. Hence $0 \theta 0 \sigma = \sigma 0 \theta = \sigma \in (\Pi_2)$. Since $0 \sigma F_G \subset \sigma F_G \implies \sigma 0 \sigma F_G \subset \sigma F_G \implies 0 \theta 0 \sigma F_G \subset 0 \sigma 0 F_G$. Again $0 \sigma F_G \subset \sigma 0 \sigma F_G \implies 0 \sigma F_G \subset 0 \sigma 0 F_G \implies 0 \sigma 0 \sigma F_G \subset 0 \sigma F_G$. Hence $0 \theta 0 \sigma = \sigma 0 \theta = \sigma \in (\Pi_2)$. Further, since $0 \sigma 0 \sigma = \sigma 0 \theta 0 \sigma = (0 \sigma 0 \sigma 0 \sigma)/(0 \sigma 0 \sigma) = (0 \sigma 0 \sigma 0 \sigma)/(0 \sigma 0 \sigma) = (0 \sigma 0 \sigma 0 \sigma)/(0 \sigma 0 \sigma) = \theta 0 \sigma 0 \sigma \theta = \theta 0 \sigma 0 \sigma \theta = (0 \sigma 0 \sigma 0 \sigma)/(0 \sigma 0 \sigma) = (0 \sigma 0 \sigma 0 \sigma)/(0 \sigma 0 \sigma) = (0 \sigma 0 \sigma 0 \sigma)/(0 \sigma 0 \sigma) = \theta 0 \sigma 0 \sigma \theta = \theta 0 \sigma 0 \sigma \theta \implies \theta 0 \sigma 0 \sigma \theta \implies (\Pi_2)$. Again any alternating products of $k \geq 5$ factors is equal to another such product of $(k - 2)$ factor and the statement holds for it.

**Theorem 4.24.** If $\theta \in (\Pi_2), (\Pi_4)$ and $\sigma \in (\Pi_2), (\Pi_3)$, then $\pi \in (\Pi_2)$ if $\pi$ is any product of the factors $\theta$ and $\sigma$.

**Proof.** If $\theta \in (\Pi_2), (\Pi_4)$ and $\sigma \in (\Pi_2), (\Pi_3)$, then $0 \sigma F_G \subset \sigma F_G, 0 \sigma F_G \subset \sigma 0 \sigma F_G$, and $0 F_G \subset \sigma 0 F_G$ for any $F_G \subset F_\varepsilon$. The proof follows from theorem 4.23.

Note: Let $(F_\varepsilon, \mu)$ be a SGTS. Clearly the soft mappings $\theta = i_\mu \in (\Pi_2), (\Pi_4)$ and $\sigma = c_\mu \in (\Pi_2), (\Pi_3)$, by theorem 4.24 any product of factor $i_\mu$ and $c_\mu$ is idempotent. In particular $i_\mu c_\mu i_\mu c_\mu = i_\mu c_\mu$ and $c_\mu i_\mu c_\mu i_\mu = c_\mu i_\mu$ so that any product of this kind is equal to one of the mappings $i_\mu, c_\mu, i_\mu c_\mu, c_\mu i_\mu, i_\mu c_\mu i_\mu, c_\mu i_\mu c_\mu$.

**Theorem 4.25.** Let $(F_\varepsilon, \mu)$ be a SGTS. Then the soft mappings $i_\mu, c_\mu, i_\mu c_\mu, c_\mu i_\mu, i_\mu c_\mu i_\mu, c_\mu i_\mu c_\mu$ are all belong to $(\Pi_2)$. 


**Proof.** Take $\pi_1 = i_\mu$ and $\pi_2 = c_\mu$, where $i_\mu F_\mu$ be the soft $\mu$-interior of the soft set $F_\mu$ and $c_\mu F_\mu$ be the soft $\mu$-closure of the soft set $F_\mu$ w.r.t. the SGT $\mu$. Clearly the soft mappings $\pi_1 = i_\mu \in (\Pi 2), (\Pi 4)$ and $\pi_2 = c_\mu (\Pi 2), (\Pi 3)$. So by theorem 4.24, the soft mappings $i_\mu, c_\mu, i_\mu c_\mu, c_\mu i_\mu, i_\mu c_\mu i_\mu, c_\mu i_\mu c_\mu$ are all belong to $(\Pi 2)$. 

**Definition 4.26.** Let $(F_\mu, \mu)$ be a SGTS. Then a soft set $F_\mu \subset F_\mu$ is said to be a soft $\mu$-semi-open set iff $F_\mu \subset c_\mu i_\mu F_\mu$ (i.e., the case when $\pi = c_\mu i_\mu$). The class of all soft $\mu$-semi-open sets is denoted by $\delta_{(\mu)}$ or $\delta_\mu$.

**Definition 4.27.** Let $(F_\mu, \mu)$ be a SGTS. Then a soft set $F_\mu \subset F_\mu$ is said to be a soft $\mu$-pre-open set iff $F_\mu \subset i_\mu c_\mu F_\mu$ (i.e., the case when $\pi = i_\mu c_\mu$). The class of all soft $\mu$-pre-open sets is denoted by $\rho_{(\mu)}$ or $\rho_\mu$.

**Definition 4.28.** Let $(F_\mu, \mu)$ be a SGTS. Then a soft set $F_\mu \subset F_\mu$ is said to be a soft $\mu$-open set iff $F_\mu \subset i_\mu c_\mu F_\mu$ (i.e., the case when $\pi = i_\mu c_\mu$). The class of all soft $\mu$-open sets is denoted by $\alpha_{(\mu)}$ or $\alpha_\mu$.

**Definition 4.29.** Let $(F_\mu, \mu)$ be a SGTS. Then a soft set $F_\mu \subset F_\mu$ is said to be a soft $\mu$-$\alpha$-open set iff $F_\mu \subset i_\mu c_\mu c_\mu F_\mu$ (i.e., the case when $\pi = i_\mu c_\mu c_\mu$). The class of all soft $\mu$-$\alpha$-open sets is denoted by $\beta_{(\mu)}$ or $\beta_\mu$.

**Example 4.30.** Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu = \{F_\emptyset, F_A, F_\emptyset\}$ where $F_A = \{(e_1, \{h_3\}), (e_2, \{h_1\})\}$. Then $(F_\emptyset, \mu)$ is a SGTS. The soft set $F_\emptyset = \{(e_1, \{h_1, h_3\}), (e_2, \{h_1\})\}$ is a soft $\mu$-semi-open sets

**Example 4.31.** Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu = \{F_\emptyset, F_\emptyset, F_\emptyset\}$ where $F_B = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_3\})\}$. Then $(F_\emptyset, \mu)$ is a SGTS. The soft sets $F_\emptyset = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2\})\}$, $F_H = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2\})\}$ are soft $\mu$-pre-open sets

**Example 4.32.** Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu = \{F_\emptyset, F_\emptyset, F_\emptyset\}$ where $F_D = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$. Then $(F_\emptyset, \mu)$ is a SGTS. The soft sets $F_D = \{(e_1, \{h_1, h_2\}), (e_2, \{h_2\})\}$ is a soft $\mu$-$\alpha$-open sets

**Example 4.33.** Let $U = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1\}$ and $\mu = \{F_\emptyset, F_\emptyset, F_\emptyset, F_\emptyset\}$ where $F_P = \{(e_1, \{h_1\}), F_Q = \{(e_1, \{h_1\})\}$, $F_R = \{(e_1, \{h_1, h_2\})\}$, $F_S = \{(e_1, \{h_1, h_3, h_4\})\}$. Then $(F_\emptyset, \mu)$ is a SGTS. The soft set $F_G = \{(e_1, \{h_1, h_3, h_4\})\}$ is a soft $\mu$-$\beta$-open sets

**Theorem 4.34.** Let $(F_\mu, \mu)$ be a SGTS. Then $\delta_\mu, \rho_\mu, \alpha_\mu$ and $\beta_\mu$ are SGT’s.

**Proof.** Follows from theorem 4.5.

Now consider the soft mappings $\pi \in (\Pi 5)$

**Theorem 4.35.** If $\pi \in (\Pi 5)$, then every soft $\pi$-open set is soft $\pi^n$-open and $\pi F_\mu \subset \pi^n F_\mu$ for $n \in N$.

**Proof.** Suppose $F_\mu$ is soft $\pi$-open. Then $F_\mu \subset \pi F_\mu$. Now $F_\mu \subset \pi F_\mu \Rightarrow \pi^n F_\mu \subset \pi^{n+1} F_\mu \Rightarrow F_\mu \subset \pi F_\mu \subset \pi^n F_\mu \Rightarrow F_\mu$ is soft $\pi^n$-open.

**Theorem 4.36.** If $\pi \in (\Pi 5)$, then $\pi^n F_\mu \subset \pi F_\mu$ and soft $\pi$-open sets and soft $\pi^n$-open sets coincide, $n \in N$. 

Proof. Suppose \( \pi \in (\Pi 5) \), then \( \pi^2 F_G \subseteq \pi F_G \Rightarrow \pi^{m+1} F_G \subseteq \pi^m F_G \) and \( \pi^n F_G \subseteq \pi F_G \). Hence by theorem 4.35, \( \pi F_G = \pi^0 F_G \). Therefore soft \( \pi \)-open sets and soft \( \pi^0 \)-open sets coincide.

**Theorem 4.37.** If \( 0, \sigma \in (\Pi 5) \) satisfies \( 0\sigma F_G \subseteq \sigma F_G \), then any product of factors \( 0 \) and \( \sigma \) belong to (\( \Pi 5 \)). If both \( 0 \) and \( \sigma \) occur among the factors of a product of this kind, then

1. soft \( 0\pi'\sigma \)-open \( \Rightarrow \) soft \( 0\sigma \)-open
2. soft \( 0\pi'\sigma \)-open \( \Rightarrow \) soft \( 0\sigma \)-open
3. soft \( \sigma\pi'\sigma \)-open \( \Rightarrow \) soft \( \sigma\sigma \)-open
4. soft \( \pi\sigma'\sigma \)-open \( \Rightarrow \) soft \( \sigma\sigma \)-open

The converse implication is true if no factor \( 0 \) is immediately followed by another such factor.

Proof. Since \( 0, \sigma \in (\Pi 5) \), by theorem 4.36, we have \( 0^n F_G \subseteq 0 F_G \) and \( \sigma^n F_G \subseteq \sigma F_G \) for \( n \in \mathbb{N} \) and since \( 0\sigma F_G \subseteq \sigma F_G \), \( (0\sigma)^n F_G \subseteq \sigma^n F_G \). Hence \( \pi \in (\Pi 5) \) if \( \pi = \sigma \). Suppose \( \pi \) is a product of factors \( 0 \) and \( \sigma \), containing at least one factor \( \sigma \). Then \( \pi \) can be written in the form \( \pi_1 \pi_2 \sigma_2 \), where \( \pi_1 \) and \( \pi_2 \) (may be empty) are products of factors \( 0 \) and \( \sigma \). Then \( \pi F_G = \pi_1 \pi_2 \sigma_2 F_G \). Since \( 0^n F_G \subseteq 0 F_G \), \( \sigma^n F_G \subseteq \sigma F_G \), \( (0\sigma)^n F_G \subseteq \sigma G \) by theorem 4.35, each group of factors \( \sigma \) can be replaced by \( \sigma \), by the repeated application of the theorem 4.35, we have \( \pi F_G \subseteq \sigma F_G \). Hence \( \pi \in (\Pi 5) \). Consider (1). Suppose \( F_G \) is soft \( 0\pi'\sigma \)-open, where \( \pi' \) is any product of both the factors \( 0 \) and \( \sigma \). Then \( F_G \subseteq 0\pi'\sigma F_G \). Since \( \pi F_G \subseteq 0 F_G \), \( \pi F_G \subseteq 0\pi'\sigma F_G \). By theorem 4.35, we can write \( 0\pi'\sigma F_G \subseteq (0\sigma)^n F_G \). Hence \( F_G \subseteq 0\sigma\pi F_G \Rightarrow F_G \) is soft \( 0\sigma\pi \)-open. Conversely assume that \( F_G \) is soft \( 0\pi'\sigma \)-open and no factor \( 0 \) is followed by another one in \( \pi = \sigma \pi' \). Then \( F_G \subseteq 0\sigma\pi F_G \Rightarrow (0\sigma)^n F_G \subseteq 0\sigma\pi F_G \). Hence \( F_G \subseteq 0\sigma\pi F_G \Rightarrow F_G \) is soft \( 0\pi'\sigma \)-open.

Consider (2). Suppose \( F_G \) is soft \( 0\pi'\sigma \)-open, where \( \pi' \) is any product of both the factors \( 0 \) and \( \sigma \). Then \( F_G \subseteq 0\pi'\sigma F_G \). Since \( (0\sigma)^n F_G \subseteq 0 F_G \) and \( \sigma \in (\Pi 5) \), we can write \( \sigma\pi'\sigma F_G \subseteq (0\sigma)^n F_G = \sigma (0\sigma)^n F_G \subseteq \sigma F_G \) for some \( k \in \mathbb{N} \). Hence \( F_G \subseteq 0\sigma\pi F_G \Rightarrow F_G \) is soft \( \sigma \pi'\sigma \)-open. Conversely assume that \( F_G \) is soft \( 0\pi'\sigma \)-open and no factor \( 0 \) is followed by another one in \( \pi = \sigma \pi' \). Then \( F_G \subseteq 0\sigma\pi F_G \Rightarrow (0\sigma)^n F_G \subseteq 0\sigma\pi F_G \). Hence \( F_G \subseteq 0\sigma\pi F_G \Rightarrow F_G \) is soft \( 0\pi'\sigma \)-open.

Consider (3). Suppose \( F_G \) is soft \( \sigma\pi'\sigma \)-open, where \( \pi' \) is any product of both the factors \( 0 \) and \( \sigma \). Then \( F_G \subseteq \sigma\pi'\sigma F_G \). By theorem 4.35, we can write \( (0\sigma)^n F_G \subseteq 0 F_G \). Hence \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow (0\sigma)^n F_G \subseteq 0\sigma\pi'\sigma F_G \). Hence \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow F_G \) is soft \( 0\pi'\sigma \)-open. Conversely assume that \( F_G \) is soft \( \sigma\pi'\sigma \)-open and no factor \( 0 \) is followed by another one in \( \pi = \sigma\pi' \). Then \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow (0\sigma)^n F_G \subseteq 0\sigma\pi'\sigma F_G \). Hence \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow F_G \) is soft \( 0\pi'\sigma \)-open.

Consider (4). Suppose \( F_G \) is soft \( \sigma\pi'\sigma \)-open, then \( F_G \subseteq \sigma\pi'\sigma F_G \). Since \( (0\sigma)^n F_G \subseteq 0 F_G \), we can write \( \sigma\pi'\sigma F_G \subseteq (0\sigma)^n F_G \subseteq \sigma\pi'\sigma F_G \). Since \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow F_G \) is soft \( 0\pi'\sigma \)-open. Conversely assume that \( F_G \) is soft \( 0\pi'\sigma \)-open and no factor \( 0 \) is followed by another one in \( \pi = \sigma\pi' \). Then \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow (0\sigma)^n F_G \subseteq 0\sigma\pi'\sigma F_G \). Hence \( F_G \subseteq \sigma\pi'\sigma F_G \Rightarrow F_G \) is soft \( 0\pi'\sigma \)-open.
another one in $\pi = \sigma'\pi$. Then $F_G \subset \sigma_0\sigma_F \Rightarrow (\sigma_0)\sigma_0\sigma_F \subset (\sigma_0)\sigma_0\sigma_F = (\sigma_0)^{\text{m}}\sigma_0\sigma_F$. Hence $\sigma_0\sigma_F \subset (\sigma_0)\sigma_0\sigma_F$. Since $\sigma_0\sigma_F \subset \sigma_0\sigma_F$, it is easy to show that $(\sigma_0)^{\text{m}}\sigma_0\sigma_F \subset \sigma'\pi\sigma_0\sigma_F$. Hence $F_G \subset \pi'\sigma_0\sigma_F \Rightarrow F_G$ is soft $\pi'\sigma_0$-open.

**Theorem 4.38.** If $0 \in (\Pi 4)$ and $\sigma \in (\Pi 5)$ then the statements of theorem 4.37 are valid; moreover, soft $\theta\sigma$–open $\Leftrightarrow$ (soft $\theta\sigma$-open and soft $\sigma\theta$-open) $\Rightarrow$ (soft $\theta\sigma$-open or soft $\sigma\theta$-open) $\Rightarrow$ soft $\theta\sigma$-open $\Rightarrow$ soft $\sigma\theta$-open.

**Proof.** If $0 \in (\Pi 4)$, then $\theta\theta F_G \subset F_G \Rightarrow \theta\theta\theta F_G \subset \theta\theta F_G \Rightarrow 0 \in (\Pi 5)$ and also $\sigma_0\sigma_F \subset \sigma_0\sigma_F$. Now the hypotheses of theorem 4.37 are fulfilled. Further, $\sigma_0(\theta\theta F_G) \subset \sigma_0 F_G$ and $\theta\sigma F_G \subset \sigma_0\sigma F_G$; i.e $F_G$ is soft $\theta\sigma$–open $\Rightarrow F_G \subset \theta\sigma\theta F_G \subset \theta\sigma F_G$ and $F_G \subset \sigma_0\sigma F_G \subset \sigma_0\sigma F_G = F_G$ is both soft $\theta\sigma$-open and soft $\sigma\theta$-open. Conversely assume that $F_G$ is both soft $\theta\sigma$-open and soft $\sigma\theta$-open. Then $F_G \subset \sigma_0\sigma F_G$ and $F_G \subset \sigma_0\sigma F_G \Rightarrow F_G \subset \sigma_0\sigma F_G \Rightarrow F_G \subset \sigma_0 F_G \subset \sigma_0\sigma F_G \subset \sigma_0 F_G \Rightarrow F_G$ is soft $\theta\sigma$-open. Again, $F_G$ is soft $\theta\sigma$-open or soft $\sigma\theta$-open $\Rightarrow F_G \subset \theta\theta F_G$ or $F_G \subset \sigma_0 F_G \Rightarrow F_G \subset \sigma_0 F_G \subset \sigma_0\sigma F_G$ respectively. Hence $F_G \subset \sigma_0 F_G$ by $0 \in (\Pi 4)$ $\Rightarrow F_G$ is soft $\sigma\theta$-open. And $F_G$ is soft $\theta\sigma$-open $\Rightarrow F_G \subset \sigma_0 F_G \subset \sigma_0 F_G \subset \sigma_0\sigma F_G \Rightarrow F_G$ is both soft $\theta\sigma$-open and soft $\sigma\theta$-open.

Note: Let $(F_e, \mu)$ be a SGTS. Then we can say that a soft set $F_G \subset \text{soft } i_\mu c_\mu i_\mu$ is $\pi F_G = \pi F_G$.

**Theorem 4.39.** If $\pi \in (\Pi 5)$ and $F_G \subset \text{soft } \pi F_G$ then $c_\pi* F_G = \pi F_G$.

**Proof.** Since $\pi \in (\Pi 5)$, $\pi F_G \subset \pi F_G \Rightarrow \pi F_G$ is soft $\pi^*$-closed by theorem 4.19. If $F_H \supset F_G$ is soft $\pi^*$-closed, then $F_H \supset \pi F_G \supset \pi F_G$. Hence $\pi F_G \supset F_G$ is the smallest soft $\pi^*$-closed super set of $F_G$. Hence $c_\pi* F_G = \pi F_G$.

**Theorem 4.40.** For any $\pi \in \Pi$ and $F_G \subset F_e$, we have $i_\pi F_G \subset F_G \cap \pi F_G$.

**Proof.** Suppose $F_H \subset F_G$ is soft $\pi$-open. Then $F_H \subset \pi F_H \subset \pi F_G$ so that $F_H \subset F_G \cap \pi F_G$. Hence $i_\pi F_G \subset F_G \cap \pi F_G$.

**Theorem 4.41.** Let $(F_e, \mu)$ be a SGTS and if $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$, then $i_\pi F_G = F_G \cap \pi F_G$ for any $F_G \subset F_e$.

**Proof.** Clearly $i_\pi F_G \subset c_\mu i_\mu F_G$ for $F_G \subset F_e$ and $i_\pi F_G \subset c_\mu i_\mu F_G \Rightarrow i_\mu i_\pi F_G \subset i_\mu c_\mu i_\mu F_G \Rightarrow i_\mu F_G \subset i_\mu c_\mu i_\mu F_G$. Therefore $i_\mu F_G \subset F_G \cap c_\mu i_\mu F_G$ and $i_\mu F_G \subset F_G \cap i_\mu c_\mu i_\mu F_G$. Hence $i_\mu F_G \subset \pi F_G \cap c_\mu i_\mu F_G$ and $i_\mu F_G \subset \pi F_G \cap i_\mu c_\mu i_\mu F_G$. Clearly $i_\mu F_G \subset \pi F_G \cap c_\mu i_\mu F_G$ and $i_\mu F_G \subset \pi F_G \cap i_\mu c_\mu i_\mu F_G$. Hence $i_\mu F_G \subset \pi F_G \cap i_\mu c_\mu i_\mu F_G$. Therefore $i_\mu F_G \subset \pi F_G \cap i_\mu c_\mu i_\mu F_G$. Hence $i_\mu F_G \subset \pi F_G \cap i_\mu c_\mu i_\mu F_G$. Hence $i_\pi F_G = F_G \cap \pi F_G$ for $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$. Thus $F_G \cap \pi F_G \subset i_\pi F_G$ in these two cases. But by theorem 4.40, $i_\pi F_G \subset F_G \cap \pi F_G$. Hence $i_\pi F_G = F_G \cap \pi F_G$ for $\pi = c_\mu i_\mu$ or $\pi = i_\mu c_\mu i_\mu$.

**Theorem 4.42.** For $\pi \in \Pi$ and $F_G \subset F_e$, $i_\pi F_G = F_G \cap \pi F_G$ for $F_G \subset F_e$ is true iff $c_\pi F_G = F_G \cup \pi F_G$.

**Proof.** Suppose $i_\pi F_G = F_G \cap \pi F_G$ is true. Then by theorem 4.17, $c_\pi F_G = (i_\pi)* F_G = [i_\pi(F_G)]^c = [(F_G)^c \cap \pi F_G]^c = F_G \cup \pi^* F_G \cup \pi F_G$. Conversely, suppose that $c_\pi F_G = F_G \cup \pi F_G$. Then $i_\pi F_G = (c_\pi)* F_G = [c_\pi(F_G)]^c = (F_G^c \cup \pi^*(F_G)]^c = F_G \cap [\pi^*(F_G)]^c = F_G \cap \pi F_G$. by theorem 4.17.
Theorem 4.43. Let \((F_E, \mu)\) be a SGTS. Then \(c_\pi F_G = F_G \cup \pi^*F_G\) is true if \(\pi = c_\mu i_\mu\) or \(\pi = i_\mu c_\mu i_\mu\).

Proof. The proof follows from theorem 4.41 and 4.42.

Conclusion

In the present work, we mainly study some interesting properties of the soft mapping \(\pi : S(U)_E \rightarrow S(U)_E\) which satisfy the condition \(\pi F_B \subset \pi F_D\) whenever \(F_B \subset F_D \subset F_E\). The concept of soft \(\pi\)-open set is introduced and established some of their properties. The notions of soft interior and soft closure are generalized using these sets and under suitable conditions on \(\pi\), the soft \(\pi\)-interior \(i_\pi F_G\) and the soft \(\pi\)-closure \(c_\pi F_G\) of a soft set \(F_G \subset F_E\) are easily obtained by explicit formulas. We expect that results in this paper will be a basis for applications of soft \(\pi\)-open sets in soft set theory and will promote the further study on soft generalized topology to carry out general frame work for the applications in practical life.

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References


